A NATURAL DEDUCTION SYSTEM PRESERVING FALSITY[1]

Wagner de Campos Sanz
Dept. of Philosophy/UFG/Brazil
sanz@fchf.ufg.br

Abstract

This paper presents a natural deduction system preserving falsity. This new system will provide us with means of reviewing a criticism made several years ago by Prior, directed against the semantical definition of logical constants by introduction and elimination rules.

Keywords: natural deduction; falsity; falsity preservation; logical constants.

If the preservation of truth is seen as one of the adequacy criteria for deductions in natural deduction systems, it seems that another equally good should be the preservation of falsity. In fact it would seem completely sound to say that a false proposition follows from other false propositions.

We would like to state the following system of natural deduction rules preserving falsity:

Introductions	$\dfrac{A_1 \qquad A_2}{A_1 \vee A_2}\,\vee i$	$\dfrac{\begin{matrix}\Gamma, [A_1]^i \\ \nabla \\ A_2\end{matrix}}{A_1 \not\subset A_2}{}^{i}\,\not\subset i$	* $\dfrac{\begin{matrix}\Gamma \\ \nabla \\ A\end{matrix}}{\exists \chi A[\alpha/\chi]}\,\exists i$
	$\dfrac{A_1}{A_1 \wedge A_2}\,l\wedge i \qquad \dfrac{A_2}{A_1 \wedge A_2}\,r\wedge i$		$\dfrac{A[\alpha/\tau]}{\forall x A[\alpha/\chi]}\,\forall i$
Eliminations	$\dfrac{A_1 \vee A_2}{A_1}\,l\vee e \qquad \dfrac{A_1 \vee A_2}{A_2}\,r\vee e$	$\dfrac{A_1 \quad A_1 \not\subset A_2}{A_2}\,\not\subset e$	$\dfrac{\exists \chi A[\alpha/\chi]}{A[\alpha/\tau]}\,\forall e$
	$\dfrac{A_1 \wedge A_2 \quad \begin{matrix}\Gamma_1, [A_1]^i \\ \nabla \\ C\end{matrix} \quad \begin{matrix}\Gamma_2, [A_2]^i \\ \nabla \\ C\end{matrix}}{C}{}^{i}\,\wedge e$		* $\dfrac{\forall \chi A[\alpha/\chi] \quad \begin{matrix}\Gamma, [A]^i \\ \nabla \\ C\end{matrix}}{C}{}^{i}\,\forall e$

Energeia ISSN 1666-5732 v.4 nº 1-2 pp.215-220 2005

Each "deduction" rule above should be read as follows: a false conclusion would follow if all immediate premises were false and/or every subsidiary deduction preserved falsity. A subsidiary deduction preserves falsity when the subsidiary conclusion of the subsidiary derivation would also be necessarily false, if all hypotheses were false.

Here, in this system, suppositions are working as suppositions of falsity. In the above rules the symbol α represents an individual parameter, τ represents an individual term and χ represents an individual variable, A_1, A_2 and C are first-order formulas. As usual, top-formulas in square brackets are being discharged. The symbol ∇ represents a subsidiary deduction. Rules *∃i* and *∀e* are further restricted: parameter α must not occur in Γ or C. Notice that this system does not contain rules for implication, but, on the other hand, it contains introduction and elimination rules for another related propositional constant. The formula $A_1 \not\subset A_2$ could be read as **A_1 "disfollows"**[2] **from A_2**. If we call such a constant **desimplication**, then desimplication introduction rule tell us that we should conclude that $A_1 \not\subset A_2$ **is false, when falsity is preserved from** Γ, A_1 to A_2.

As can be verified, from a purely structural perspective, the above rules are identical to the rules of the minimal system for truth preservation. Their difference lies in that conjunction ($\wedge$) and disjunction ($\vee$), on the one hand, and existential ($\exists$) and universal ($\forall$), on the other, have changed places. Implication introduction and implication elimination are also structurally identical with desimplication introduction and desimplication elimination. In that way, any structural property that holds for the system preserving truth will also carry over to the system preserving falsity. Confluence, normalization and consistency are valid for this new system. One just chooses the proof one likes most and adapts it to the present case (see for example [Prawitz]). Notice also that consistency in the new system means that it is not true that every formula can be falsified or, better still, not every formula can be refuted. Also, any formula derived from no hypothesis at all should be regarded as a refuted formula.

For an intuitive interpretation of our system we could read it as an elucidation of logical relations holding between counterfactuals. In other words, if A is derived from Γ in our system, in such a way that all propositions in Γ are counterfactuals, then A would be a counterfactual. Notice that the usual system for truth preservation cannot accurately represent a counterfactual inference:

$$\frac{\text{Bizet was a fellow citizen of Verdi} \qquad \text{(false)}}{\text{Bizet was a fellow citizen of Verdi} \vee \text{Bizet was a fellow citizen of Zola.} \qquad \text{(true)}}\, i\vee$$

In the system for falsity preservation, from the same premise we could obtain in a sound way another false conclusion:

$$\frac{\text{Bizet was a fellow citizen of Verdi} \quad \text{(false)}}{\text{Bizet was a fellow citizen of Verdi} \wedge \text{Bizet was a fellow citizen of Zola.} \quad \text{(false)}}\ i\wedge$$

We notice that among the above rules one probably would not receive approval from intuitionists, the *∀e*. Nonetheless, that would be quite surprising, if we also consider that the corresponding *∃e* rule in the system preserving truth is structurally identical with *∀e*. However, we also notice that the entire propositional fragment, in the above system, is intuitionistically acceptable.

In the system preserving falsity, a derivation of *A* from *Γ* cannot be read as saying that *A* follows deductively from *Γ*. Now the concept of deductive consequence is working bottom-up, from *A* to *Γ*, which could seem awkward, but is acceptable. Just notice that refutability is usually a notion derived from the concept of deductive consequence, as being transmitted backwards in the chain of deductive consequences. It has been thus defined in [Curry, Chap. 6].

In fact, any formula *A* will be refutable, when another formula *B* is refutable and, at the same time, *B* follows deductively from *A*. In this way if one presents a clausal definition of the concept of deductive consequence for such a new system it could de done as follows, where $\Gamma,\Delta,\Delta_1,\Delta_2$ are lists of formulas:

(i) $A \Rightarrow A$
(ii) in case $A_1 \Rightarrow \Gamma$ *and* $A_2 \Rightarrow \Delta$, *then* $A_1 \vee A_2 \Rightarrow \Gamma,\Delta$
(iii) in case $A_2 \Rightarrow \Gamma, A_1$, *then* $A_1 \not\subset A_2 \Rightarrow \Gamma$
(iv) in case $A \Rightarrow \Gamma$ *and parameter a doesn't occur in* Γ, *then* $\exists x A[a/x] \Rightarrow \Gamma$
(v) in case $A_1 \Rightarrow \Gamma$ *or* $A_2 \Rightarrow \Gamma$, *then* $A_1 \wedge A_2 \Rightarrow \Gamma$
(vi) in case $A[a/\tau] \Rightarrow \Gamma$, *then* $\forall x A[a/x] \Rightarrow \Gamma$
(vii) in case $C \Rightarrow \Gamma, A_1$ *then* $C \Rightarrow \Gamma, A_1 \vee A_2$
(viii) in case $C \Rightarrow \Gamma, A_2$, *then* $C \Rightarrow \Gamma, A_1 \vee A_2$
(ix) in case $C \Rightarrow \Gamma, A_2$ *and* $A_1 \Rightarrow \Delta$, *then* $C \Rightarrow \Gamma, \Delta, A_1 \not\subset A_2$
(x) in case $C \Rightarrow \Gamma, A[a/\tau]$, *then* $C \Rightarrow \Gamma, \exists x A[a/x]$
(xi) in case $C \Rightarrow \Gamma, A_1$ *and* $C \Rightarrow \Delta, A_2$, *then* $C \Rightarrow \Gamma, \Delta, A_1 \wedge A_2$
(xii) in case $C \Rightarrow \Gamma, A$ *such that parameter a doesn't occur in* Γ *and C, then* $C \Rightarrow \Gamma, \forall x A[a/x]$.

On analyzing clauses *(ii)-(vi)*, with the exception of clause *(iii)*, it can be seen that they are basically clauses for introducing logical constants on the left of a sequent.

However, as regards sequent calculus there is a major difference. It lies in the number of formula occurrences at each side of symbol $\Rightarrow$. Turning now to clause *(iii)*, it can be observed again that its logical operator is usually absent from sequent calculus. But, on closer examination, it can be noticed that $\not\subset$ left introduction is similar to $\supset$ right introduction. Finally, clauses *(vii)-(xii)* correspond to elimination rules and are presented as right introductions. As is to be expected, $\not\subset$ right introduction is similar to $\supset$ left introduction.

As the next step, a treatment of negation could be added to the new system in the same way we use to define negation in the system preserving truth, i.e., by means of the absurd ($\bot$): $\sim A \equiv_{df} A \supset \bot$. For the system preserving falsity we just change logical operators by its corresponding partners: $\sim A \equiv_{df} A \not\subset T$. The symbol *T* represents tautology and to deal with it we add another rule to the system, a rule structurally similar to the *ex falso quodlibet*. Let's call it *ex truth quodlibet*:

$$\frac{T}{A}\ eT$$

We regard it as an elimination rule, just as much as the *ex false quodlibet* is regarded as an elimination rule in the intuitionist system for truth preservation. Now, after introducing tautology, it is also possible to handle implication by means of an explicit definition: $A \supset B \equiv_{df} (A \not\subset T) \vee B$.

Last but not least, to obtain a system corresponding to the classical system for truth preservation, one can just add any one of a list of equivalent alternatives to the principle of indirect deduction used in the system for truth preservation. For example, you may add *double negation cancelation*:

$$\frac{\sim\sim A}{A}\ dnc$$

In the literature we found some partial natural deduction systems that also preserve falsity, for example [Tamminga; and Tamminga & Tanaka's]. It would seem worthwhile to reproduce here an interesting result concerning propositional calculus.

For any propositional variable there will always be as many interpretations making it false, as there are interpretations making its negation false. So, any formula in the system for falsity preservation, derived from a list of hypotheses, which is empty or contains only propositional variables and negations of propositional variables, but not both for the same propositional variable, will be a falsifiable formula. In other words, it certainly will not be tautology (another system with the same purpose, although somewhat cumbersome, can be seen in [Caicedo]).

We believe that the system preserving falsity allows us to throw some light on the problem of the meaning of natural deduction systems. We claim that any radical syntactical interpretation trying to establish the meaning of a logical constant by means of its introduction and elimination rules is untenable. As there is no structural difference between conjunction/disjunction rules, universal/existential rules and implication/unimplication rules, respectively, in the systems for truth and falsity preservation, then, before we can decide which constant is being elucidated by the given rules, we should establish beforehand if the system is preserving truth or falsity. As they are structurally identical, such information cannot be given syntactically. In fact such a result holds even for axiomatic systems.

Some time ago Prior contested the view that introduction and elimination rules could be considered an effective and complete means for defining logical constants. His argument consisted in showing introduction and elimination rules for a “fictitious constant”, but, as the critics rightly observed, his pair of rules was disharmonious. However, after defining a system preserving falsity, we believe that Prior’s objection could be taken up again, at least partially. Introduction and elimination rules cannot define completely what a logical constant means, because it is necessary to know beforehand which criterion is being applied to the rules: preservation of truth or preservation of falsity. These criteria are independent of natural deduction rules and external to them. With a system for falsity preservation at hand, it can be observed that any derived formula may be read as a demonstrated formula or as a refuted formula. This depends on the interpretation given to the logical constants. There is no need to read $\wedge$ as conjunction. It can be read as disjunction, etc. So, here it seems, we cannot distinguish syntactically which interpretation is being presumed.

Notes

[1] Our investigations were initially motivated by an exposition by R. Ertola at the *XXV Encontro Brasileiro de Lógica*, held at Unicamp-Brazil in 2003, where the author developed an inverse reading of natural deduction rules as falsity preservation.

[2] Obviously the word *disfollows* is a neologism and it means that A_1 *doesn’t follow from* A_2.

Bibliography

Caicedo, X. (1978). "A Formal System for the Non-Theorems of the Propositional Calculus". Notre Dame Journal of Formal Logic, v. XIX, nº 1, pp.147-151.

Curry, H.B. (1963). *Foundations of Mathematical Logic*. Dover.

Gentzen, G. (1969). "Investigations Into Logical Deduction". M.E. Szabo, ed. *Gentzen Collected Papers*. North-Holland, pp.68-131.

Prawitz, D. (1965). *Natural Deduction*. Almqvist&Wiksell.

Prior, A.N. (1967). "The Runabout Inference-Ticket". P.F. Strawson, ed. *Philosophical Logic*. Oxford, pp.129-31.

Tamminga, A.M. (1994). "Logics of Rejection: Two Systems of Natural Deduction". Logique et Analyse, 146, pp.169-208.

Tamminga, A.M. & K.Tanaka (1999). "A Natural Deduction System for First Degree Entailment". Notre Dame Journal of Formal Logic, v. 40, nº 2, pp.258-272.